Supplementary document for the module galIMF.py

Based on C. Schulz, J. Pflamm-Altenburg, and P. Kroupa (2015 A&A) and Yan, Z., Jerabkova, T., Kroupa, P. (2017 A&A)

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This document develops the mathematical forms needed for optimally sampling (Schulz et al 2015) object masses from a 3-part-power-law stellar initial mass function (IMF) and a single-part-power-law embedded cluster mass function (ECMF). Optimal sampling (Kroupa et al. 2013) is a deterministic sampling technique from a distribution function such that the result has no Poisson uncertainty.

This document serves as supplementary documentation for the galIMF module written in the Python 3 programming language. The galIMF module calculates the galaxy-wide stellar initial mass function based on the IGIMF theory (Yan et al. 2017) and optimal sampling. All equations used in the galIMF module are derived in this file. For example, in the Part IMF, equations 1 to 10 are from Yan et al (2017) and Schulz et al (2015) while other equations are derivations for different cases that are useful for the computer code.

The symbols used here are similar as in the code (i.e. M_L in the code is written M_L here) but different from the publications. We summarize the meaning of the symbols before the derivation.

Equations in the galIMF.py module are labeled with the same equation numbers here, allowing the user to go through the code and get oriented in exactly what each function computes.

The assumptions or the starting point of the derivations are given in Yan et al. (2017) and Schulz et al. (2015). We strongly recommend to go through these two papers before using the galIMF module. A galaxy-wide IMF grid covering different metallicities ([Fe/H]) and star formation rates (SFR) computed using the galIMF module can be found in Jerabkova et al. (2017, in prep.).

References are not mentioned or fully given for some equations and parameter values in this document, for more information, please go through Yan et al. (2017).

Part I. IMF

1. Symbols

All mass in this file and galIMF.py module are in solar mass unit.

M_L and M_U are the theoretical lower and upper stellar star mass limits, respectively. They serve as integration bounds.

We divide the mass range between M_L and M_U into $N+2$ mass ranges (M_L–m_{N+1}, m_{N+1}–m_N, m_N–m_{N-1}...m_{i+1}–m_i...m_1–m_0) to sample N stars, with each mass range generating one star, except for the first and last mass range, i.e., M_L–m_{N+1} and m_1–m_0 does not generate a star.

We set $m_0 = M_U$ and we also define $m_1 \equiv M_{max}$. The use of two names for the same meaning is because the computer code needs numbered name (m_0, m_1...m_{N+1}), while equations and descriptions are easier to read with meaningful subscript (L, U or max) rather than numbers.

M_{max} (also as m_1) is the most-massive-star's upper integration limit. Note that this limit is not M_U because there is no star generated in the mass range M_{max} to M_U (also as m_1–m_0).

M_1 is the star generated in mass range m_2–m_1. It is the most massive sampled star. M_2 is the star generated in mass range m_3–m_2 and so on.

Note that symbols with lower case letter, m, are all integration limits, while generated stellar masses are indicated by M_i where i is a positive integer. Capital M with letter subscript are special names for a few integration limits.

In summary $M_U \equiv m_0 > M_{max} \equiv m_1 > M_1 > m_2 > M_2 > m_3 > M_3 > ... > m_N > M_N > m_{N+1} > M_L > m_{N+2}$, where $m_{N+1} \approx M_L$.

The default values in galIMF.py are:
$M_L = 0.08$ (hydrogen burning mass limit),
$M_{turn} = 0.5$ and $M_{turn,2} = 1$ is where the IMF slope changes,
$M_U = 150$ (WK conjecture, Weidner & Kroupa 2004).

We note, in below, the capital N is the total number of stars in a star cluster, while the small n is a positive integer defined for fast calculation with no physical meaning.

2. Derivation

The IMF is:

$$\xi_{star} = \begin{cases} k_1 M^{-\alpha_1}, & \text{if } M_L < M < M_{turn}, \\ k_2 M^{-\alpha_2}, & \text{if } M_{turn} < M < M_{turn,2}, \\ k_3 M^{-\alpha_3}, & \text{if } M_{turn,2} < M < M_U, \end{cases} \tag{1}$$

with

$$\begin{aligned} k_1 &= k_2 M_{turn}^{\alpha_1-\alpha_2}, \\ k_2 &= k_3 M_{turn,2}^{\alpha_2-\alpha_3} \\ &= k_3, \text{ if } M_{turn,2} = 1, \end{aligned} \tag{2}$$

where $M_{turn} = 0.5$, $M_{turn,2} = 1$, $\alpha_1 = 1.3$, $\alpha_2 = 2.3$ are parameters for canonical IMF (Kroupa 2001).

$$\alpha_3 = \begin{cases} 2.3, & x < -0.87, \\ -0.41x + 1.94, & x \geq -0.87, \end{cases} \tag{3}$$

with $x = -0.14[\mathrm{Fe/H}] + 0.99 \log_{10}(\rho_{cl}/(10^6 M_\odot \mathrm{pc}^{-3}))$.

The total number of stars is:

$$N = \int_{M_L}^{M_{max}} \xi(M)\,\mathrm{d}M. \tag{4}$$

The total mass of the stellar population is:

$$M_{ecl} = M_{tot} = \int_{M_L}^{M_{max}} M\xi(M)\,\mathrm{d}M. \tag{5}$$

Note that the upper integration limit of the above two equations are M_{max} rather than M_U. The reason is mentioned above in the Section Symbols.

Divide both integrals in equation 4 and 5 into $N+1$ separate integrals, each integral representing one individual star except for the first one that is redundant:

$$\begin{aligned} N = &\int_{m_{N+1}}^{m_N} \xi(M)\,\mathrm{d}M + \int_{m_N}^{m_{N-1}} \xi(M)\,\mathrm{d}M + ... \\ &+ \int_{m_{i+1}}^{m_i} \xi(M)\,\mathrm{d}M + ... + \int_{m_2}^{m_1} \xi(M)\,\mathrm{d}M, \end{aligned} \tag{6}$$

$$\begin{aligned} M_{tot} = &\int_{m_{N+1}}^{m_N} M\xi(M)\,\mathrm{d}M + \int_{m_N}^{m_{N-1}} M\xi(M)\,\mathrm{d}M + ... \\ &+ \int_{m_{i+1}}^{m_i} M\xi(M)\,\mathrm{d}M + ... + \int_{m_2}^{m_1} M\xi(M)\,\mathrm{d}M, \end{aligned} \tag{7}$$

where $m_1 = M_{max}$.

Each integral must give one star:

$$1 = \int_{m_{i+1}}^{m_i} \xi(M)\,\mathrm{d}M, \tag{8}$$

with the resulting stellar mass being:

$$M_i = \int_{m_{i+1}}^{m_i} M\xi(M)\,\mathrm{d}M. \tag{9}$$

And the redundant mass, M_r, that does not form a star is:

$$1 > \int_{M_L}^{m_{N+1}} \xi(M)\,\mathrm{d}M, \tag{10}$$

$$M_L > M_r = \int_{M_L}^{m_{N+1}} M\xi(M)\,\mathrm{d}M. \tag{11}$$

The optimal sampling normalization condition is:

$$I = \int_{m_1}^{M_U} \xi(M)\,\mathrm{d}M = \int_{M_{max}}^{M_U} \xi(M)\,\mathrm{d}M. \tag{12}$$

This equation defines M_{max} for a given M_U. $I = 1$ is the canonical normalization condition. It is possible to change I to match the observed M_1-M_{ecl} relation, where M_1 is the most massive stellar mass in the star cluster.

2.1. For $M_{turn,2} < M_{max}$

Inserting equation 1 to 12 for $\alpha_3 \neq 1$,

$$I = k_3 \int_{M_{max}}^{M_U} M^{-\alpha_3}\,\mathrm{d}M = \frac{k_3}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{max}^{1-\alpha_3}). \tag{13}$$

From this,

$$k_3 = \frac{I(1-\alpha_3)}{M_U^{1-\alpha_3} - M_{max}^{1-\alpha_3}}. \tag{14}$$

We consider only the case $\alpha_3 \approx \alpha_2 > \alpha_1 > 1$ as indicated by observations.

For simplicity, consider only $\alpha_2 \neq 2$ and $\alpha_3 \neq 2$. The case $\alpha_{2,3} = 2$ can be approximated by $\alpha_{2,3} = 2 + \Delta$, where $\Delta \ll 1$, if needed.

Inserting equation 1 into equation 5,

$$\begin{aligned}
&M_{tot} \\
&= \int_{M_L}^{M_{turn}} k_1 M^{1-\alpha_1}\,\mathrm{d}M + \int_{M_{turn}}^{M_{turn,2}} k_2 M^{1-\alpha_2}\,\mathrm{d}M + \int_{M_{turn,2}}^{M_{max}} k_3 M^{1-\alpha_3}\,\mathrm{d}M, \\
&= \frac{k_1}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - M_L^{2-\alpha_1}) + \frac{k_2}{2-\alpha_2}(M_{turn,2}^{2-\alpha_2} - M_{turn}^{2-\alpha_2}) + \frac{k_3}{2-\alpha_3}(M_{max}^{2-\alpha_3} - M_{turn,2}^{2-\alpha_3}), \\
&= \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3} M_{turn}^{\alpha_1-\alpha_2}}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - M_L^{2-\alpha_1}) + \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3}}{2-\alpha_2}(M_{turn,2}^{2-\alpha_2} - M_{turn}^{2-\alpha_2}) \\
&+ \frac{k_3}{2-\alpha_3}(M_{max}^{2-\alpha_3} - M_{turn,2}^{2-\alpha_3}).
\end{aligned} \tag{15}$$

Inserting equation 14 into equation 15,

$$\begin{aligned}&\frac{M_{tot}M_U^{1-\alpha_3}}{I(1-\alpha_3)}-\frac{M_{turn,2}^{\alpha_2-\alpha_3}M_{turn}^{\alpha_1-\alpha_2}}{2-\alpha_1}(M_{turn}^{2-\alpha_1}-M_L^{2-\alpha_1})\\&-\frac{M_{turn,2}^{\alpha_2-\alpha_3}}{2-\alpha_2}(M_{turn,2}^{2-\alpha_2}-M_{turn}^{2-\alpha_2})+\frac{M_{turn,2}^{2-\alpha_3}}{2-\alpha_3}\\&=\frac{M_{max}^{2-\alpha_3}}{2-\alpha_3}+\frac{M_{tot}M_{max}^{1-\alpha_3}}{I(1-\alpha_3)}.\end{aligned}\tag{16}$$

This give us $M_{max}=M_{max}(M_{tot},I,M_L,M_{turn},M_{turn,2},M_U,\alpha_1,\alpha_2,\alpha_3)$.

M_{tot} is given by the stellar mass of the cluster M_{ecl}.

Inserting equation 14 into equation 1,

$$\xi_{star}=\xi(M;M_{ecl},I,M_L,M_{turn},M_{turn,2},M_U,\alpha_1,\alpha_2,\alpha_3).\tag{17}$$

With equation 1 and equation 8 every m_i can now be calculated:

$$m_{i+1}=\begin{cases}\left(m_i^{1-\alpha_3}-\frac{1-\alpha_3}{k_3}\right)^{\frac{1}{1-\alpha_3}}\\ \text{, if } M_{turn,2}<m_{i+1}<m_i<M_U,\\ \left(M_{turn,2}^{1-\alpha_2}+\frac{k_3}{k_2}\frac{1-\alpha_2}{1-\alpha_3}(m_i^{1-\alpha_3}-M_{turn,2}^{1-\alpha_3})-\frac{1-\alpha_2}{k_2}\right)^{\frac{1}{1-\alpha_2}}\\ \text{, if } M_{turn}<m_{i+1}<M_{turn,2}<m_i<M_U,\\ \left(m_i^{1-\alpha_2}-\frac{1-\alpha_2}{k_2}\right)^{\frac{1}{1-\alpha_2}}\\ \text{, if } M_{turn}<m_{i+1}<m_i<M_{turn,2},\\ \left(M_{turn}^{1-\alpha_1}+\frac{k_2}{k_1}\frac{1-\alpha_1}{1-\alpha_2}(m_i^{1-\alpha_2}-M_{turn}^{1-\alpha_2})-\frac{1-\alpha_1}{k_1}\right)^{\frac{1}{1-\alpha_1}}\\ \text{, if } M_L<m_{i+1}<M_{turn}<m_i<M_{turn,2},\\ \left(m_i^{1-\alpha_1}-\frac{1-\alpha_1}{k_1}\right)^{\frac{1}{1-\alpha_1}}\\ \text{, if } M_L<m_{i+1}<m_i<M_{turn},\end{cases}\tag{18}$$

$$m_{i+n}=\left(m_i^{1-\alpha}-n\cdot\frac{1-\alpha}{k}\right)^{\frac{1}{1-\alpha}},\tag{19}$$

where n is a positive integer, $\alpha=\alpha_1$ when $m_{i+1}<m_i<M_{turn}$, $\alpha=\alpha_2$ when $M_{turn}<m_{i+1}<m_i<M_{turn,2}$ and $\alpha=\alpha_3$ when $M_{turn,2}<m_{i+1}<m_i<M_U$. With, e.g., $n=100$, this equation calculates m_{i+100}, thus we know 100 stars are generated with mass between m_i and m_{i+100} without calculating m_{i+1}, $m_{i+2}...m_{i+99}$. As, in large star clusters with a great number of stars, these 100 stars will result in very similar masses, it is then not necessary to calculate an accurate mass for each of them but an average mass to represent them all.

The mass of each star, M_i, is determined by equation 9. Keep in mind that we consider only the case $\alpha \neq 2$, then the stellar mass of all the stars with a mass smaller than m_i and larger than m_{i+n} is:

$$n \cdot \bar{M} = M_i + M_{i+1} + ... + M_{i+n-1} =$$
$$\begin{cases} \frac{k_1}{2-\alpha_1}(m_i^{2-\alpha_1} - m_{i+n}^{2-\alpha_1}), & \text{if } M_L < m_{i+n} < m_i < M_{turn}, \\ \frac{k_1}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - m_{i+n}^{2-\alpha_1}) + \frac{k_2}{2-\alpha_2}(M_i^{2-\alpha_2} - M_{turn}^{2-\alpha_2}), & \text{if } M_L < m_{i+n} < M_{turn} < m_i < M_{turn,2}, \\ \frac{k_2}{2-\alpha_2}(m_i^{2-\alpha_2} - m_{i+n}^{2-\alpha_2}), & \text{if } M_{turn} < m_{i+n} < m_i < M_{turn,2}, \\ \frac{k_2}{2-\alpha_2}(M_{turn,2}^{2-\alpha_2} - m_{i+n}^{2-\alpha_2}) + \frac{k_3}{2-\alpha_3}(M_i^{2-\alpha_3} - M_{turn,2}^{2-\alpha_3}), & \text{if } M_{turn} < m_{i+n} < M_{turn,2} < m_i < M_U, \\ \frac{k_3}{2-\alpha_3}(m_i^{2-\alpha_3} - m_{i+n}^{2-\alpha_3}), & \text{if } M_{turn,2} < m_{i+n} < m_i < M_U, \end{cases} \tag{20}$$

with $\bar{M}$ being the average mass of these n stars.

The special case of $m_{i+n} < M_L$ needs to be considered separately as the above equation would generate stars with masses smaller than M_L, which is not allowed by our definition of M_L (see section Symbols).

When the program finds $m_{i+n} < M_L$, it will reset n to a smaller value n' such that $m_{i+n'+1} < M_L < m_{i+n'}$. Then we have:

$$n' \cdot \bar{M} \approx \frac{k_1}{2-\alpha_1}(m_i^{2-\alpha_1} - M_L^{2-\alpha_1}), \text{ if } m_{i+n} < M_L < m_i < M_{turn}. \tag{21}$$

2.2. For $M_{turn} < M_{max} < M_{turn,2}$

If the cluster mass is small enough, $M_{max} < M_{turn,2}$ is possible. This happens when one sets the lowest possible star cluster mass $M_{ecl,L}$ to be smaller than 2.72 solar masses.

Similar to equation 13, inserting equation 1 into equation 12 for $\alpha_2 \neq 1$:

$$\begin{aligned} I &= k_2 \int_{M_{max}}^{M_{turn,2}} M^{-\alpha_2}\,\mathrm{d}M + k_3 \int_{M_{turn,2}}^{M_U} M^{-\alpha_3}\,\mathrm{d}M, \\ &= \frac{k_2}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{max}^{1-\alpha_2}) + \frac{k_3}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}), \\ &= \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3}}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{max}^{1-\alpha_2}) + \frac{k_3}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}). \end{aligned} \tag{22}$$

From this we get

$$k_3 = I \cdot \left[\frac{M_{turn,2}^{\alpha_2-\alpha_3}}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{max}^{1-\alpha_2}) + \frac{1}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3})\right]^{-1}. \tag{23}$$

For $\alpha_2 \neq 2$:

$$
\begin{aligned}
M_{tot} &= \int_{M_L}^{M_{turn}} k_1 M^{1-\alpha_1}\,\mathrm{d}M + \int_{M_{turn}}^{M_{max}} k_2 M^{1-\alpha_2}\,\mathrm{d}M, \\
&= \frac{k_1}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - M_L^{2-\alpha_1}) + \frac{k_2}{2-\alpha_2}(M_{max}^{2-\alpha_2} - M_{turn}^{2-\alpha_2}), \\
&= \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3} M_{turn}^{\alpha_1-\alpha_2}}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - M_L^{2-\alpha_1}) + \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3}}{2-\alpha_2}(M_{max}^{2-\alpha_2} - M_{turn}^{2-\alpha_2}).
\end{aligned}
\tag{24}
$$

Inserting equation 23 gives M_{max}:

$$
\begin{aligned}
&\frac{M_{max}^{2-\alpha_2}}{2-\alpha_2} + \frac{M_{tot}}{I(1-\alpha_2)} M_{max}^{1-\alpha_2} \\
=&\frac{M_{tot} M_{turn,2}^{1-\alpha_2}}{I(1-\alpha_2)} + \frac{M_{tot} M_{turn,2}^{\alpha_3-\alpha_2}}{I(1-\alpha_3)}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}) \\
&- \frac{M_{turn}^{\alpha_1-\alpha_2}}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - M_L^{2-\alpha_1}) + \frac{M_{turn}^{2-\alpha_2}}{2-\alpha_2},
\end{aligned}
\tag{25}
$$

where m_{i+1} is the same as equation 18 but using the new k_3 in equation 23.

Finally, the mass of stars is the same as in equation 20

2.3. For $M_{max} < M_{turn}$

This happens when the lowest possible star cluster mass, $M_{ecl,L}$, is set to be even smaller below a realistic value. Nevertheless, we consider this case to make the code more robust.

Inserting equation 1 into equation 12 for $\alpha_2 \neq 1$:

$$
\begin{aligned}
I &= k_1 \int_{M_{max}}^{M_{turn}} M^{-\alpha_1}\,\mathrm{d}M + k_2 \int_{M_{turn}}^{M_{turn,2}} M^{-\alpha_2}\,\mathrm{d}M + k_3 \int_{M_{turn,2}}^{M_U} M^{-\alpha_3}\,\mathrm{d}M, \\
&= \frac{k_1}{1-\alpha_1}(M_{turn}^{1-\alpha_1} - M_{max}^{1-\alpha_1}) + \frac{k_2}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{turn}^{1-\alpha_2}) \\
&+ \frac{k_3}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}), \\
&= \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3} M_{turn}^{\alpha_1-\alpha_2}}{1-\alpha_1}(M_{turn}^{1-\alpha_1} - M_{max}^{1-\alpha_1}) + \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3}}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{turn}^{1-\alpha_2}) \\
&+ \frac{k_3}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}).
\end{aligned}
\tag{26}
$$

From this we get k_3

$$
\begin{aligned}
I \cdot k_3^{-1} =&\frac{M_{turn,2}^{\alpha_2-\alpha_3} M_{turn}^{\alpha_1-\alpha_2}}{1-\alpha_1}(M_{turn}^{1-\alpha_1} - M_{max}^{1-\alpha_1}) \\
&+ \frac{M_{turn,2}^{\alpha_2-\alpha_3}}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{turn}^{1-\alpha_2}) + \frac{1}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}).
\end{aligned}
\tag{27}
$$

And then

$$\begin{aligned} M_{tot} &= \int_{M_L}^{M_{max}} k_1 M^{1-\alpha_1}\,\mathrm{d}M = \frac{k_1}{2-\alpha_1}(M_{max}^{2-\alpha_1} - M_L^{2-\alpha_1}), \\ &= \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3} M_{turn}^{\alpha_1-\alpha_2}}{2-\alpha_1}(M_{max}^{2-\alpha_1} - M_L^{2-\alpha_1}). \end{aligned} \tag{28}$$

Inserting the new k_3 into equation 27 gives M_{max}:

$$\begin{aligned} &\frac{M_{max}^{2-\alpha_1}}{2-\alpha_1} + \frac{M_{tot}}{I(1-\alpha_1)} M_{max}^{1-\alpha_1} \\ =&\frac{M_{tot} M_{turn}^{1-\alpha_1}}{I(1-\alpha_1)} + \frac{M_{tot} M_{turn}^{\alpha_2-\alpha_1}}{I(1-\alpha_2)}(M_{turn,2}^{1-\alpha_2} - M_{turn}^{1-\alpha_2}) \\ &+ \frac{M_{tot} M_{turn,2}^{\alpha_3-\alpha_2} M_{turn}^{\alpha_2-\alpha_1}}{I(1-\alpha_3)}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}) + \frac{M_L^{2-\alpha_1}}{2-\alpha_1}, \end{aligned} \tag{29}$$

where m_{i+1} is the same as equation 18 but using the new k_3 in equation 27.

Finally, the total mass in stars is the same as in equation 20.

Part II.
ECMF

The ECMF is a single-part-power-law function which is similar to the case of canonical IMF which is a 3-part-power-law function. In fact, it is possible for us to skip the derivation for the case of the ECMF and use the same equations in Part I with $\alpha_1 = \alpha_2 = \alpha_3$. However, for the sake of computational efficiency we consider the ECMF case separately.

3. Symbols

Same as Part I but replace ”star” with ”star cluster”.
All masses are in the unit of solar mass.
The default values in the code are $M_L = 5$ and $M_U = 10^9$.

4. Derivation

$$\xi = kM^{-\beta}, \tag{30}$$

$$N = \int_{M_L}^{M_{max}} \xi(M)\,\mathrm{d}M, \tag{31}$$

$$SFR * \delta t = M_{tot} = \int_{M_L}^{M_{max}} M\xi(M)\,\mathrm{d}M. \tag{32}$$

Divide both integrals in equation 31 and equation 32 into $N+1$ separate integrals, each integral representing one individual star cluster except for the first one that is redundant:

$$\begin{aligned} N = &\int_{m_{N+1}}^{m_N} \xi(M)\,\mathrm{d}M + \int_{m_N}^{m_{N-1}} \xi(M)\,\mathrm{d}M + ... \\ &+ \int_{m_{i+1}}^{m_i} \xi(M)\,\mathrm{d}M + ... + \int_{m_2}^{m_1} \xi(M)\,\mathrm{d}M, \end{aligned} \tag{33}$$

$$\begin{aligned} M_{tot} = &\int_{m_{N+1}}^{m_N} M\xi(M)\,\mathrm{d}M + \int_{m_N}^{m_{N-1}} M\xi(M)\,\mathrm{d}M + ... \\ &+ \int_{m_{i+1}}^{m_i} M\xi(M)\,\mathrm{d}M + ... + \int_{m_2}^{m_1} M\xi(M)\,\mathrm{d}M, \end{aligned} \tag{34}$$

where $m_1 = M_{max}$ and $m_{N+1} \approx M_L$.

Each integral gives one star cluster:

$$1 = \int_{m_{i+1}}^{m_i} \xi(M)\,\mathrm{d}M, \tag{35}$$

with the star cluster mass being:

$$M_i = \int_{m_{i+1}}^{m_i} M\xi(M)\,\mathrm{d}M. \tag{36}$$

And the redundant mass, M_r, is:

$$1 > \int_{M_L}^{m_{N+1}} \xi(M)\,\mathrm{d}M, \tag{37}$$

$$M_L > M_r = \int_{M_L}^{m_{N+1}} M\xi(M)\,\mathrm{d}M. \tag{38}$$

The optimally sampling normalization condition is:

$$I = \int_{m_1}^{m_0} \xi(M)\,\mathrm{d}M = \int_{M_{max}}^{M_U} \xi(M)\,\mathrm{d}M. \tag{39}$$

Inserting equation 30 into equation 39 for $\beta \neq 1$, we have:

$$I = k \int_{M_{max}}^{M_U} M^{-\beta}\,\mathrm{d}M = \frac{k}{1-\beta}(M_U^{1-\beta} - M_{max}^{1-\beta}). \tag{40}$$

Thus

$$k = \frac{I(1-\beta)}{M_U^{1-\beta} - M_{max}^{1-\beta}}. \tag{41}$$

Consider only the case $\beta > 1$ for the following derivations as indicated by observation. If $M_U = \infty$ then

$$k = \frac{I(\beta - 1)}{M_{max}^{1-\beta}}. \tag{42}$$

For $\beta \neq 2$:

Inserting equation 30 into equation 32

$$M_{tot} = k \int_{M_L}^{M_{max}} M^{1-\beta} \, \mathrm{d}M = \frac{k}{2-\beta}(M_{max}^{2-\beta} - M_L^{2-\beta}). \tag{43}$$

Inserting equation 41 into equation 43

$$M_{tot} = I \cdot \frac{1-\beta}{2-\beta} \cdot \frac{M_{max}^{2-\beta} - M_L^{2-\beta}}{M_U^{1-\beta} - M_{max}^{1-\beta}}. \tag{44}$$

For $\beta = 2$:

Equation 41 and 42 reduced to

$$k = \frac{I}{M_{max}^{-1} - M_U^{-1}}, \tag{45}$$

and

$$k = IM_{max}, \tag{46}$$

respectively.

Inserting equation 30 into equation 32:

$$M_{tot} = k \int_{M_L}^{M_{max}} M^{-1} \, \mathrm{d}M = k(\ln M_{max} - \ln M_L). \tag{47}$$

Inserting equation 45 into equation 47:

$$M_{tot} = I \cdot \frac{\ln M_{max} - \ln M_L}{M_{max}^{-1} - M_U^{-1}}. \tag{48}$$

Equations 44 and 48 yield

$$M_{max} = M_{max}(M_{tot}, I, M_U, M_L, \beta). \tag{49}$$

$M_{tot} = SFR \cdot \delta t$, with a fixed $\delta t \approx 10$ Myr (see Yan et al. 2017) and an assumed SFH. Thus M_{max} is determined by:

$$M_{max} = M_{max}(SFR, \delta t, I, M_U, M_L, \beta). \tag{50}$$

Inserting equation 41 and equation 50 into equation 30

$$\xi = \xi(M; SFR, \delta t, I, M_U, M_L, \beta). \tag{51}$$

Then the mass of each cluster generated can be calculated from equations 35 and 36.

Using equations 30, 35 and 41 every m_i can be calculated:

$$m_{i+n} = \left(m_i^{1-\beta} - n \cdot \frac{1-\beta}{k} \right)^{\frac{1}{1-\beta}}, \tag{52}$$

for $i > 0$, where n is a positive integer.

Then with equations 30, 36 and 41 every M_i can be calculated.
If $n > 1$, the average cluster mass $\bar{M}$ is,
for $\beta \neq 2$:

$$n \cdot \bar{M} = M_i + M_{i+1} + ... + M_{i+n-1} = \frac{k}{2-\beta} \cdot (m_i^{2-\beta} - m_{i+n}^{2-\beta}), \tag{53}$$

for $\beta = 2$:

$$n \cdot \bar{M} = M_i + M_{i+1} + ... + M_{i+n-1} = k(\ln m_i - \ln m_{i+n}). \tag{54}$$